't Hooft GR Exercise 14.1

We first work out $d\omega_3^2$ using (14.9):

$$d\omega_2^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$$

$$d\omega_3^2 = d\chi^2 + \sin^2\chi\left[d\theta^2 + \sin^2\theta\, d\varphi^2\right]$$

Now we compute $d\omega^2 = B(\rho)\,d\rho^2 + \rho^2\left[d\theta^2 + \sin^2\theta\, d\varphi^2\right]$,

with $B = \dfrac{1}{1-\frac{1}{2}\lambda\rho^2}$, $\rho = \sqrt{\frac{2}{\lambda}}\sin\chi$.

$$\Rightarrow \quad \rho^2 = \frac{2}{\lambda}\sin^2\chi, \quad B = \frac{1}{1-\cancel{\frac{1}{2}}\cancel{\lambda}\frac{\cancel{2}}{\cancel{\lambda}}\sin^2\chi} = \frac{1}{1-\sin^2\chi}$$

$$\Rightarrow d\omega^2 = \frac{1}{1-\sin^2\chi}\,d\rho^2 + \frac{2}{\lambda}\sin^2\chi\left[d\theta^2 + \sin^2\theta\, d\varphi^2\right]$$

$$\rho = \sqrt{\frac{2}{\lambda}}\sin\chi \quad \Rightarrow \quad d\rho = \sqrt{\frac{2}{\lambda}}\cos\chi\, d\chi,$$

$$d\omega^2 = \frac{1}{\cancel{1-\sin^2\chi}}\frac{2}{\lambda}\cancel{\cos^2\chi}\,d\chi^2 + \frac{2}{\lambda}\sin^2\chi\left[d\theta^2 + \sin^2\theta\, d\varphi^2\right]$$

$$\Rightarrow \quad \frac{\lambda}{2}\,d\omega^2 = d\chi^2 + \sin^2\chi\left[d\theta^2 + \sin^2\theta\, d\varphi^2\right]$$

Make ~~coordinate~~ global scale transformation $\omega \to \omega' = \sqrt{\frac{2}{\lambda}}\,\omega$,

$$\Rightarrow d\omega' = \sqrt{\frac{2}{\lambda}}\,d\omega$$

The above equation applies to w' as well:

$$\frac{\lambda}{2}(dw')^2 = d\psi^2 + \sin^2\psi\,[d\theta^2 + \sin^2\theta\, d\varphi^2]$$

$$\frac{\cancel{\lambda}}{\cancel{2}}\frac{\cancel{2}}{\cancel{\lambda}}\, dw^2 = d\psi^2 + \sin^2\psi\,[d\theta^2 + \sin^2\theta\, d\varphi^2]$$

$$\boxed{= dw_3^2}$$

This is the 3-sphere metric as we computed at the start.

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